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Preface to the Republication of "Uniformly High Order Essentially Non-oscillatory Schemes, III," by Harten, Engquist, Osher, and Chakravarthy

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The classic paper by Harten, Engquist, Osher, and Chakravarthy on ENO schemes [1] has had a tremendous influence on research in numerical solutions of hyperbolic conservation laws since its publication.

The original and beautiful idea of this paper is a uniformly high-order interpolation recipe with an adaptive stencil, termed ENO (Essentially Non-Oscillatory) reconstruction. It is well known that the wider the stencil, the higher the order of accuracy of the interpolation, provided the function being interpolated is smooth inside the stencil. Traditional finite difference methods are based on fixed stencil interpolations. For example, to obtain an interpolation for cell i to third-order accuracy, the information of the three cells $i - 1$, i, and $i + 1$ can be used. This works well for globally smooth problems. The resulting scheme is linear for linear PDEs; hence stability can be analyzed by Fourier transforms. However, fixed stencil interpolation of second- or higher-order accuracy is necessarily *oscillatory* near a discontinuity. Such oscillations (called the Gibbs phenomenon in spectral methods) do not decay in magnitude when the mesh is refined. It is a nuisance to say the least for practical calculations, and often leads to numerical instabilities in nonlinear problems containing discontinuities.

Before 1987, there were two common ways to eliminate or reduce such spurious oscillations near discontinuities. One way was to add an artificial viscosity. This could be tuned so that it was large enough near the discontinuity to suppress, or at least to reduce, the oscillations, but was small elsewhere to maintain high-order accuracy. One disadvantage of this approach is that fine tuning of the parameter controlling the size of the artificial viscosity is problem dependent. Another way was to apply limiters to eliminate the oscillations. In effect, one reduced the order of accuracy of the interpolation near the discontinuity (e.g., using a linear rather than a quadratic interpolant near the shock). By carefully designing such limiters, the TVD (total variation diminishing) property could be achieved for nonlinear scalar one-dimensional problems. One disadvantage of this approach is that accuracy necessarily degenerated to first order near *smooth* extrema.

The ENO idea proposed in [1] is the first successful attempt to obtain a uniformly high-order accurate, yet essentially non-oscillatory, interpolation (i.e., the magnitude of the oscillations decay as $O(h^r)$, where r is the order of accuracy) for piecewise smooth functions. The generic solution for hyperbolic conservation laws is in the class of piecewise smooth functions. The reconstruction in [1] is a natural extension of an earlier second-order version of Harten and Osher [2]. In [1], Harten, Engquist, Osher, and Chakravarthy investigated different ways of measuring local smoothness to determine the local stencil, and developed a hierarchy that begins with one or two cells, then adds one cell at a time to the stencil from the two candidates on the left and right, based on the size of the two relevant Newton divided differences. This seems to be the most robust way for a wide range of grid sizes, h, both before and inside the asymptotic regime.

As one can see from the numerical examples in [1] and in later papers, ENO schemes are indeed uniformly high-order accurate and resolve shocks with sharp and monotone (to the eye) transitions. ENO schemes are especially suitable for problems containing both shocks and complicated smooth flow structures, such as occur in shock interactions with turbulent flow.

This paper of Harten, Engquist, Osher, and Chakravarthy [1] has been cited 144 times, according to the ISI database. The original authors and many other researchers have followed the pioneer work of [1], improving the methodology and expanding the area of its applications. ENO schemes based on point values and TVD Runge–Kutta time discretizations, which can save computational costs

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significantly for multi-space dimensions, were developed in [3, 4]. Later, biasing in the stencil choosing process to enhance stability and accuracy were developed in [5, 6]. Weighted ENO (WENO) schemes were developed, using a linear combination of all candidate stencils instead of just one as in the original ENO [7, 8]. ENO schemes based on other than polynomial building blocks were constructed in [9, 10]; subcell resolution and artificial compressibility to sharpen contact discontinuities were studied in [11, 12, 4]. Multidimensional ENO schemes based on general triangulation were developed in [13]. ENO schemes for Hamilton–Jacobi-type equations were designed in [14, 15, 16]. Combination of ENO with multiresolution ideas was pursued in [17]. On the application side, ENO has been successfully used to simulate shock turbulence interactions [4, 18, 19]; to the direct simulation of turbulence [18, 20]; to relativistic hydrodynamics equations [21]; to shock vortex interactions and other gas dynamics problems [22, 23]; to incompressible flow problems [24, 25]; to semiconductor device simulation [5, 26]; etc. This list may be incomplete and biased by my own research experience, but one can see that most of the problems solved by ENO schemes are of the type in which solutions contain both strong shocks and rich smooth region structures. Lower-order methods usually have difficulties for such problems and it is thus attractive and efficient to use a high-order stable method such as ENO to handle them.

Today the study and application of ENO schemes are still very active. We expect ENO schemes and the basic methodology to be even more successful in the future.

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